Near-unanimity terms are decidable

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Def. An n-ary operation f on a set A is a near-unanimity operation, if $n \geq 3$ and for all $x, y \in A$

$$f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, \ldots, x, y) = x.$$

Typical example: lattices, algebras with lattice reducts:

$$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

If a finite algebra $\mathbf{A}$ has a near-unanimity term operation, then

- the variety $\mathcal{V}(\mathbf{A})$ is congruence-distributive
- the variety $\mathcal{V}(\mathbf{A})$ is finitely axiomatizable
- the clone $\mathrm{Clo}(\mathbf{A})$ is finitely generated
- the relational clone $\mathrm{Inv}(\mathrm{Clo}(\mathbf{A}))$ is finitely generated
- the constraint satisfaction problem for $\mathbf{A}$ is tractable
- the algebra $\mathbf{A}$ admits a natural duality

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Natural duality problem:

Input: finite algebra **A** with finite signature
Problem: decide whether **A** admits a natural duality

Thm. (Davey, Heindorf and McKenzie, 1995)
An algebra generating a congruence distributive variety admits a natural duality if and only if it has a near-unanimity term operation.

Near-unanimity problem:

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Problem: decide whether **A** has a near-unanimity term operation

The set of n-ary operations in the clone $\mathrm{Clo}\,\mathbf{A}$ can be easily computed, but the arity of the near-unanimity operation is unknown

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Let $\mathbf{A}$ be an algebra and $\mathbf{B} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(2)$ be the two-generated free algebra. Then the following are equivalent:

- $\mathbf{A}$ has a near-unanimity term operation
- $\mathbf{B}$ has a term operation that is a near-unanimity operation on the generating set $\{x, y\}$ of $\mathbf{B}$
- The subalgebra of $\mathbf{B}^\omega$ generated by the tuples

$$\begin{aligned} \bar{y}_0 &= \langle y, x, x, \ldots \rangle \\ \bar{y}_1 &= \langle x, y, x, \ldots \rangle \\ \bar{y}_2 &= \langle x, x, y, \ldots \rangle \\ &\vdots \end{aligned}$$

  contains the tuple

$$\bar{x} = \langle x, x, x, \ldots \rangle$$

Thm. (McKenzie, 1997)
It is undecidable for a finite algebra $\mathbf{B}$ and two elements $x, y \in B$ whether $\mathbf{B}$ has a term operation that is a near-unanimity operation on $\{x, y\}$.

Def. A Minsky machine has two registers R_1, R_2 that can contain arbitrary natural numbers. The program is a finite set of states, containing an initial and halting state, and a finite list of commands of the form

- in state i increase the value of register R_r by one and go to state j,
- in state i if the value of register R_r is zero then go to state j, otherwise decrease the value of the register by one and go to state k.

Thm. It is undecidable for a finite algebra $\mathbf{B}$ and two elements $x, y \in B$ whether $\mathbf{B}$ has a term operation that is a near-unanimity operation on $B \setminus \{x, y\}$.

Idea:

- Minsky-machine $\mathcal{M}$ halts (finite computation) iff an algebra $\mathbf{B}(\mathcal{M})$ has a "partial" near-unanimity term operation (finite term)
- full control over the elements and operations of $\mathbf{B}(\mathcal{M})$
- almost majority operation $m(x, y, z, u)$ that needs a "key" term in place of u to unlock it
- term u is slim (forced shape), contains the halting computation of $\mathcal{M}$
- $\mathbf{B}(\mathcal{M})$ has an absorbing element to propagate local inconsistencies to the root

Motivation: Let $\mathbf{C} = \langle C; F \rangle$ be an infinite algebra of finite signature, and $X, Y \subseteq C$. Assume that there exists a congruence $\vartheta \in \mathrm{Con}\,\mathbf{C}$ such that

- ϑ has finitely many equivalence classes,
- X is a union of equivalence classes,
- the membership problem in ϑ is decidable,

then the problem of whether X and $\mathrm{Sg}_{\mathbf{C}}(Y)$ are disjoint is decidable.

Def. Let $\mathbf{C} = \langle C; F \rangle$ be an algebra. The equivalence relation ϑ is a weak congruence of $\mathbf{C}$ with respect to a monoid $E \leq \mathrm{End}\,\mathbf{C}$ of endomorphisms if

- every class of ϑ is closed under the endomorphisms in E,
- for every n-ary operation $f \in F$ and pairs $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \vartheta$ there exist endomorphisms $e_1, \ldots, e_n \in E$ such that

$$f(a_1, \ldots, a_n)\ \vartheta\ f(e_1(b_1), \ldots, e_n(b_n)).$$

Thm. The second condition also holds for arbitrary terms.

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Def. The quotient $\mathbf{C}/\vartheta$ is a multi-valued algebra on the set C/ϑ of equivalence classes where

- for each operation symbol $f \in F$ a family $\{ f_{e_1,\dots,e_n} : e_1, \dots, e_n \in E \}$ of operations are defined as

$$f_{e_1,\dots,e_n}(b_1/\vartheta, \dots, b_n/\vartheta) := f(e_1(b_1), \dots, e_n(b_n))/\vartheta$$

- the endomorphisms in E induce the identity operation on $\mathbf{C}/\vartheta$

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Application: The near-unanimity problem is decidable.

Let $\mathbf{A} = \langle A; F \rangle$ be a finite algebra, $\mathbf{B} = \mathbf{F}_{\mathcal{V}(\mathbf{A})}(2)$ the free algebra generated by $\{x, y\}$, $\mathbf{C} = \mathbf{B}^\omega$, $X = \{\langle x, x, \dots \rangle\}$, and $Y = \{ \bar{y}_i : i \in \omega \}$ where $\bar{y}_i = \langle x, \dots, x, \underset{i}{\underbrace{y}}, x, \dots \rangle$.

Def. For a permutation $\pi \in S_\omega$ let e_π be the automorphism of $\mathbf{B}^\omega$ that permutes the coordinates, i.e.

$$e_\pi(\bar{b}) = e_\pi(\langle b_0, b_1, \dots \rangle) = \langle b_{\pi(0)}, b_{\pi(1)}, \dots \rangle.$$

The weak congruence ϑ with respect to $E = \{ e_\pi : \pi \in S_\omega \}$ is defined as

$$\begin{aligned} \bar{a} \; \vartheta \; \bar{b} &\iff \bar{a} = e_\pi(\bar{b}) \text{ for some } \pi \in S_\omega \\ &\iff \text{the elements of } \bar{a} \text{ and } \bar{b} \text{ are the same with multiplicities.} \end{aligned}$$

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Def. The characteristic function of $\bar{a} \in \mathbf{B}^\omega$ is the map $\chi_{\bar{a}} : B \to \omega^+$ defined as

$$\chi_{\bar{a}}(t) = |\{\, i \in \omega : a_i = t \,\}|\,.$$

Example. For $\bar{x} = \langle x, x, \dots \rangle$ and $t \in B$

$$\chi_{\bar{x}}(t) = \begin{cases} \omega & \text{if } t(x,y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Example. For $\bar{y}_i = \langle x, \dots, x, y, x, \dots \rangle$ and $t \in B$

$$\chi_{\bar{y}_i}(t) = \begin{cases} 1 & \text{if } t(x,y) = y, \\ \omega & \text{if } t(x,y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Note: ϑ is the kernel of the map $\chi : \mathbf{B}^\omega \to (\omega^+)^B$, $\bar{a} \mapsto \chi_{\bar{a}}$.

Conditions:

- ϑ has finitely many equivalence classes: not true
- X is a union of equivalence classes: true
- the membership problem in ϑ is decidable: true
- $\mathrm{Sg}_{\mathbf{C}}(Y)$ is a union of equivalence classes: true
- the multi-valued operations of $\mathbf{C}/\vartheta$ are effectively computable: true

Take a ternary operation $f \in F$, elements $\bar{a}, \bar{b}, \bar{c} \in \mathbf{B}^\omega$, permutations $\pi, \sigma, \tau \in S_\omega$. We need to find the class of $f(e_\pi(\bar{a}), e_\sigma(\bar{b}), e_\tau(\bar{c}))$:

$$\bar{a} = \langle a_0, a_1, \dots, a_{j-1}, a_j, a_j, a_j, \dots \rangle$$
$$\bar{b} = \langle b_0, b_1, \dots\dots, b_{k-1}, b_k, b_k, b_k, \dots \rangle$$
$$\bar{c} = \langle c_0, c_1, \dots\dots\dots, c_{l-1}, c_l, c_l, c_l, \dots \rangle$$

At most $j + k + l$ columns of $e_\pi(\bar{a})$, $e_\sigma(\bar{b})$, $e_\tau(\bar{c})$ differ from $\langle a_j, b_k, c_l \rangle$, so we may assume that the permutations move only the first $j + k + l$ many elements: finitely many choices

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Def. For a fixed positive integer m let r be the automorphism of $\mathbf{B}^\omega$ that inserts m new copies of the first coordinate:

$$r(\langle b_0, b_1, b_2, \ldots \rangle) = \langle \underbrace{b_0, b_0, \ldots, b_0}_{m+1\text{-many}}, b_1, b_2, \ldots \rangle.$$

Let $E \leq \operatorname{End} \mathbf{B}^\omega$ be the monoid generated by the r and $\{ e_\pi : \pi \in S_\omega \}$, and ϑ the corresponding smallest weak congruence on $\mathbf{B}^\omega$.

The number of occurrences of an element $t \in B$ in a tuple $\bar{b} \in B$

- 0: does not occur,
- $1, \ldots, m-1$: occurs this many times modulo m,
- m: occurs m-multiple many times (at least once),
- ω occurs infinitely many times.

Def. characteristic function: $\chi_{\bar{b}} : B \to \{0, 1, \ldots, m, \omega\}$

There are finitely many characteristic functions!

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- $\mathrm{Sg}_{\mathbf{C}}(Y)$ is a union of equivalence classes: not true

For example, $r(\langle y, x, x, \ldots \rangle) = \langle \underbrace{y, \ldots, y}_{m+1\text{-many}}, x, x, \ldots \rangle \notin \mathrm{Sg}_{\mathbf{C}}(Y)$.

True, if $\mathbf{B}$ has an $m+1$-ary minority term. Can be fixed in general with an $m+1$-ary special weak near-unanimity operation:

$$w(x, x, \ldots, x) \approx x,$$
$$w(y, x, \ldots, x) \approx w(x, y, x, \ldots, x) \approx \cdots \approx w(x, \ldots, x, y),$$
$$w(w(y, x, \ldots, x), x, \ldots, x) \approx w(y, x, \ldots, x)$$

and by changing the generator set Y.

- the multi-valued operations of $\mathbf{C}/\vartheta$ are effectively computable: true, but not trivial

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$$w(w(y, x, \ldots, x), x, \ldots, x) \approx w(y, x, \ldots, x)$$

and by changing the generator set Y.

- the multi-valued operations of $\mathbf{C}/\vartheta$ are effectively computable: true, but not trivial

Conditions:

- ϑ has finitely many equivalence classes: true
- X is a union of equivalence classes: true
- the membership problem in ϑ is decidable: true
- $\mathrm{Sg}_{\mathbf{C}}(Y)$ is a union of equivalence classes: not true

For example, $r(\langle y, x, x, \ldots\rangle) = \langle \underbrace{y, \ldots, y}_{m+1\text{-many}}, x, x, \ldots\rangle \notin \mathrm{Sg}_{\mathbf{C}}(Y)$.

True, if $\mathbf{B}$ has an $m+1$-ary minority term. Can be fixed in general with an $m+1$-ary special weak near-unanimity operation:

$$w(x, x, \ldots, x) \approx x,$$
$$w(y, x, \ldots, x) \approx w(x, y, x, \ldots, x) \approx \cdots \approx w(x, \ldots, x, y),$$
$$w(w(y, x, \ldots, x), x, \ldots, x) \approx w(y, x, \ldots, x)$$

and by changing the generator set Y.

- the multi-valued operations of $\mathbf{C}/\vartheta$ are effectively computable: true, but not trivial

Thm. Let **A** be a finite algebra on an n-element set with operations of maximum arity of k. If **A** has a near-unanimity term operation, then it has one with arity at most

$$2^{2^{2^{ckn^2}}}$$

for some constant c.

Cor. It is decidable of a finite algebra in a congruence distributive (join-semi-distributive) variety whether it admits a natural duality.

Open Porblem 1. (Davey, McKenzie) Natural duality problem.

Open Problem 2. Given a finite partial algebra, decide whether it has a term that is defined on all near-unanimous assignments and satisfies the near-unanimity identities.

Open Problem 3. (Feder) Given a finite set Γ of relations on a finite set, decide whether there exists a near-unanimity operation that is compatible with each member of Γ.

Open Problem 4. (Zádori) Is there a finite set of relations on a finite set such that the clone of compatible operations is congruence distributive but contains no near-unanimity operation?

Open Problem 5. Given a finite set of operations on a finite set, decide if the clone of compatible relations is finitely generated.

Open Problem 6. Given a finite set of relations on a finite set, decide if the clone of compatible operations is finitely generated.

Open Problem 7. Given a finite set of operations and a finite set of relations on the same underlying set, decide if the functional and relational clones they generate are duals of each other.

Thm. The existence of an edge-term in a finite algebra is decidable

$$
\begin{aligned}
t(y,y,x,x,x,\dots,x) &\approx x,\\
t(y,x,y,x,x,\dots,x) &\approx x,\\
t(x,x,x,y,x,\dots,x) &\approx x,\\
t(x,x,x,x,y,\dots,x) &\approx x,\\
&\vdots
\end{aligned}
$$

Def. A finite algebra **A** has few subpowers if there is some polynomial $p(n)$ such that the

$$\log_2 |\{ B : B \text{ is a subuniverse of } \mathbf{A}^n \}| \le p(n).$$

Thm. (Berman, Idziak, Markovic, McKenzie, Valeriote, Willard, 2006) A finite algebra **A** has few subpowers if and only if it has an edge term operation.

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t(y, y, x, x, x, \dots, x) &\approx x, \\
t(y, x, y, x, x, \dots, x) &\approx x, \\
t(x, x, x, y, x, \dots, x) &\approx x, \\
t(x, x, x, x, y, \dots, x) &\approx x, \\
&\vdots
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Def. A finite algebra $\mathbf{A}$ has few subpowers if there is some polynomial $p(n)$ such that the

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Thm. (Berman, Idziak, Markovic, McKenzie, Valeriote, Willard, 2006) A finite algebra $\mathbf{A}$ has few subpowers if and only if it has an edge term operation.

Thm. (Barto, Kozik, Niven, 2007) Every finitely generated congruence distributive variety has a cyclic term, i.e. a term t satisfying the identities

$$t(x, x, \dots, x) \approx x$$
$$t(x_1, x_2, \dots, x_{n-1}, x_n) \approx t(x_2, x_3, \dots, x_n, x_1) \qquad (n \geq 2).$$

Cyclic terms are special weak near-unanimity terms.

Open Problem 8. Given a finite algebra, decide if it has a cyclic term satisfying the identities

Open Problem 9. Find a Mal'tsev condition that is undecidable for finite algebras.